

# LIFSCHITZ TAILS AND LOCALISATION FOR A CLASS OF SCHRÖDINGER OPERATORS WITH RANDOM BREATHER-TYPE POTENTIAL

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**ABSTRACT.** We derive bounds on the integrated density of states of Schrödinger operators with a random, ergodic potential. The potential depends on a sequence of random variables, not necessarily in a linear way. An example of such a random Schrödinger operator is the breather model, as introduced by Combes, Hislop and Mourre. For these models we show that the integrated density of states near the bottom of the spectrum behaves according to the so called Lifshitz asymptotics. This enables us to prove localisation in certain energy/disorder regimes.

## 1. INTRODUCTION

In this paper we study spectral properties of certain Schrödinger operators with random potential. The spectral theory of such operators has been studied since the eighties in the mathematical literature and there are several monographs devoted to this topic, see e.g. [CFKS87, CL90, PF92, Sto01]. Certain spectral features, like the non-randomness of the spectral components and the integrated density of states, are shared by a wide variety of models under mild ergodicity and regularity assumptions. However specific characteristics — like the existence of a certain spectral type — depend on the concrete model at hand.

Our aim is to establish for a class random Schrödinger operators the Lifshitz asymptotics of the integrated density of states and localisation near the bottom of the spectrum. To explain this in more detail, we will define the considered class of operators and thereafter present our results precisely.

Our results concern random Schrödinger operators  $H_\omega$  of the following type. Let  $W_{\text{per}}$  be a  $\mathbb{Z}^d$ -periodic potential whose positive part  $V_{\text{per},+} := \max(0, V_{\text{per}})$  belongs to  $L^1_{\text{loc}}$  and whose negative part  $V_{\text{per},-} := \max(0, -V_{\text{per}})$  is in the Kato class. Let  $H_\omega = H_0 + W_\omega$  and  $H_0 = -\Delta + W_{\text{per}}$ , where  $W_\omega$  is a random potential, i.e. a stochastic field, given by

$$(1) \quad W_\omega(x) := \sum_{k \in \mathbb{Z}^d} u(\lambda_k(\omega), x - k).$$

Here  $\lambda_k: \Omega \rightarrow [\lambda_-, \lambda_+]$ ,  $k \in \mathbb{Z}^d$  is a collection of non-trivial, independent, identically distributed random variables on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . The distribution of  $\lambda_0$  is denoted by  $\mu$  and we assume  $\inf \text{supp } \mu = \lambda_-$ . The function  $u$  is called *single site potential* and enjoys throughout the paper the following properties.

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**Assumption 1.** The single site potential  $u: \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  is called *monotone in the randomness* if it is jointly measurable and satisfies the following:

For all  $\lambda \in [\lambda_-, \lambda_+]$  we have

$$(2) \quad \text{supp } u(\lambda, \cdot) \subset \Lambda_1 := [-\frac{1}{2}, \frac{1}{2}]^d$$

$$(3) \quad \frac{\partial}{\partial \lambda} u(\lambda, \cdot) \in L^\infty(\Lambda_1) \quad \text{with} \quad \kappa_1 := \sup_{x \in \mathbb{R}^d} \sup_{\lambda \in [\lambda_-, \lambda_+]} \frac{\partial}{\partial \lambda} u(\lambda, x) < \infty$$

For all  $x \in \mathbb{R}^d$  and  $\lambda \in [\lambda_-, \lambda_+]$  we have

$$(4) \quad \frac{\partial u}{\partial \lambda}(\lambda, x) \geq 0$$

There exist  $\epsilon_1, \epsilon_2 > 0$  such that for all  $\lambda \in [\lambda_-, \lambda_- + \epsilon_2]$  we have

$$(5) \quad \frac{d}{d\lambda} \int_{\mathbb{R}^d} dx u(\lambda, x) \in [\epsilon_1, \epsilon_1^{-1}]$$

For all  $\lambda \in [\lambda_-, \lambda_+]$  the function

$$(6) \quad u(\lambda, \cdot) \text{ is reflection invariant with respect to all } d \text{ coordinate axes.}$$

Note that the randomness enters the potential (1) via a field of random variables  $\lambda_k, k \in \mathbb{Z}^d$ , not necessarily in a linear way. Condition (6) can be dispensed with. For this it is necessary to use in Section 3 instead of Neumann boundary conditions certain mixed boundary conditions introduced by Mezincescu in [Mez87]. Furthermore, one can assume instead of  $\text{supp } u(\lambda, \cdot) \subset \Lambda_1$  that there is a compact subset of  $\mathbb{R}^d$  which contains the support of  $u(\lambda, \cdot)$  for every  $\lambda$ . A random potential of the form (1) with a single site potential satisfying Assumption 1 gives rise to a *metrically transitive or ergodic operator*, see e.g. [Kir89] or [PF92] for the definition. In particular, there is a subset  $\Sigma$  of the real line such that the spectrum of  $H_\omega$  coincides with  $\Sigma$  almost surely.

Since  $\lambda_- = \inf \text{supp } \mu$  one can use Wely sequences to see that  $\sigma(H_{\text{per}}) = E_0$ , where  $E_0$  denotes the minimum of the almost sure spectrum of the family  $H_\omega, \omega \in \Omega$ .

**Example 1.** If we set in (1)  $u(\lambda, x - k) = \lambda f(x - k)$  we obtain an *alloy type potential*

$$(7) \quad W_\omega(x) := \sum_{k \in \mathbb{Z}^d} \lambda_k(\omega) f(x - k)$$

Such random potentials have been thoroughly studied before in the context of the Lifshitz asymptotics of the integrated density of states and localisation, see e.g. [KM83, KS86, Mez87, CH94, Sto99, Sto01, GK01, GK04, AEN<sup>+</sup>06]. If  $f$  is non-negative and sufficiently regular the resulting single site potential  $u$  is monotone in the randomness. For such alloy type models the results we are aiming at are by now well understood, therefore we will not elaborate on them further.

**Example 2.** More interesting is the situation, if we set

$$(8) \quad u(\lambda, x) = -f(\lambda x)$$

In this case the resulting stochastic field

$$(9) \quad W_\omega(x) := - \sum_{k \in \mathbb{Z}^d} f(\lambda_k(\omega)(x - k))$$

is called random *breather-type potential*, cf. [CHM96, CHN01]. If we assume for the function  $f$

$$(10) \quad \text{supp } f \subset \Lambda_{\lambda_-}, \quad f \in C_0^1(\mathbb{R}^d \setminus \{0\})$$

$$(11) \quad L^\infty(\mathbb{R}^d) \ni g(x) := -x \cdot (\nabla f)(x) \geq 0 \text{ and not identically vanishing}$$

$$(12) \quad f \text{ is reflection symmetric with respect to all coordinate axes.}$$

then the potential  $u: \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  is monotone in the randomness. Inequality (11) is called the *repulsivity property* of  $f$ .

Set

$$(13) \quad H_{\text{per}} := H_0 + V_{\text{per}} \quad \text{and} \quad V_{\text{per}}(x) := \sum_{k \in \mathbb{Z}^d} u(\lambda_-, x - k)$$

**Remark 3** (Monotonicity of the model). Condition (4) ensures that for all  $\omega \in \Omega, x \in \mathbb{R}^d, k \in \mathbb{Z}^d$  we have

$$(14) \quad u(\lambda_k, x) \geq u(\lambda_-, x) \text{ and consequently } W_\omega(x) \geq V_{\text{per}}(x)$$

Denote for any  $\lambda \in [\lambda_-, \lambda_+]$  the modified single site potential  $\tilde{u}(\lambda, x) := u(\lambda, x) - u(\lambda_-, x)$  and

$$V_\omega(x) := W_\omega(x) - V_{\text{per}}(x) = \sum_{k \in \mathbb{Z}^d} \tilde{u}(\lambda_k(\omega), x - k) \geq 0$$

hence  $H_\omega = H_{\text{per}} + V_\omega$ .

To formulate our first result we introduce some more notation. Let  $\Lambda_L(j) := [-L/2, L/2]^d + j \subset \mathbb{R}^d$  be a cube of side length  $L$  centered at  $j \in \mathbb{Z}^d$ ,  $I_L := \Lambda_L \cap \mathbb{Z}^d$  the lattice points contained in it, and  $\chi_j$  the characteristic function of  $\Lambda_1(j)$ . We write  $\Lambda_L$  for  $\Lambda_L(0)$ . For a Schrödinger operator  $H$  on  $\mathbb{R}^d$  we denote by  $H^{L,\text{per}}, H^{L,N}, H^{L,D}$  its restriction to  $\Lambda_L$  with periodic, Neumann, respectively Dirichlet boundary conditions. Since we will be mostly using Neumann b.c. we abbreviate  $H^{L,N}$  further to  $H^L$ . Let  $\chi_J(H_\omega^L)$  denote the spectral projection for the operator  $H_\omega^L$  associated with a Borel set  $J \subset \mathbb{R}$ .

An important spectral characteristic of ergodic random operators is the *integrated density of states* (IDS) denoted by  $N$ . It is defined as the limit of the distribution functions

$$(15) \quad \begin{aligned} N_\omega^L(E) &:= L^{-d} \#\{n \mid n\text{-th eigenvalue of } H_\omega^L \text{ does not exceed } E\} \\ &= L^{-d} \text{Tr} [\chi_{]-\infty, E]}(H_\omega^L)]. \end{aligned}$$

as  $L$  tends to infinity. Under our assumptions on  $H_\omega, \omega \in \Omega$ , the limit exists at all continuity points of  $E$  and is independent of  $\omega$ , for almost all  $\omega \in \Omega$ . If we replace in (15) the Neumann b.c. by periodic or Dirichlet ones, the limit distribution  $N$  does not change. Furthermore  $\inf\{E \mid N(E) > 0\} = E_0$ .

Now we are able to formulate the result on the asymptotic behaviour of the IDS at the bottom of the spectrum.

**Theorem 4** (Lifshitz Tails). *Let  $H_\omega, \omega \in \Omega$  be a random operator with potential (1) satisfying Assumption 1. Then*

$$(16) \quad \lim_{E \searrow E_0} \frac{\log |\log N(E)|}{\log(E - E_0)} \leq -\frac{d}{2}$$

One can also prove equality in the above formula (16) if one assumes not too fast decay of  $\mu([\lambda_-, \lambda_- + \epsilon])$  as  $\epsilon \rightarrow 0$ . We are only interested in the upper bound (16), which is the “hard” part of the equality, since we want to deduce spectral localisation. The bound (16) means that for  $E \searrow E_0$  asymptotically we have  $N(E) \lesssim ce^{-\tilde{c}E^{-d/2}}$ .

For breather type potentials satisfying Assumption 2 below we are furthermore able to prove localisation near the bottom of the spectrum.

Let us explain this notion. An interval  $J = [E_0, E_0 + \delta] \subset \mathbb{R}, \delta > 0$  is called a *localisation interval* for the family  $H_\omega, \omega \in \Omega$ , if  $H_\omega$  has no continuous spectrum in  $J$  and all eigenfunctions associated to eigenvalues in  $J$  are decaying exponentially, for almost all  $\omega$ . In this situation one speaks of *spectral* or *exponential localisation*. Alternatively, localisation can also be expressed in terms of the dynamics of wavepackets, in which case one speaks of *dynamical localisation*. For a detailed discussion of this notion we refer to [Sto01] or [GK04]. Although dynamical and spectral localisation are not equivalent, for the type of models considered here it turns out that as soon as the multiscale analysis applies, both versions of localisation hold, see. e.g. [DS01, Sto01, GK01, GK04].

To carry through the multiscale proof of localisation one needs to establish beforehand several a priori conditions. The first one is the so-called *initial length scale decay estimate* on the integral kernel of the resolvent of a finite box Hamiltonian. It can be derived from the Lifshitz tail asymptotics of the IDS derived in our Theorem 4. This is a well known argument established e.g. in [MH84, Klo95, KSS98, Sto01, Ves02].

The second ingredient is a *Wegner estimate*, which goes back to upper bounds on the density of states established in the paper [Weg81]. Such estimates are well known for alloy type random Schrödinger operators as considered in Example 1. Some of the first papers where such estimates have been established are [Klo95, CH94, Kir96], see also [Ves04] for a survey. In [CHM96, CHN01] it was shown that Wegner estimates hold for random breather-type potentials, as long as the following assumption holds.

**Assumption 2.** There exists an  $\epsilon_4 > 0$  such that the single site potential  $f: \mathbb{R}^d \rightarrow \mathbb{R}$  of the breather-type potential satisfies for all  $x \in \mathbb{R}^d$

$$(17) \quad L^\infty(\mathbb{R}^d) \ni -x \cdot \nabla f(x) \geq \epsilon_4 f(x) \geq 0$$

The third assumption for the multiscale analysis is *independence* of the potential values *at a sufficiently large distance*. This fact holds for the potentials considered in the present paper for the following reason: The single site potential has support in the cube  $\Lambda_1$ , for all values of  $\lambda \in [\lambda_-, \lambda_+]$ . Thus for two points  $x, y \in \mathbb{R}^d$  with  $\|x - y\|_\infty := \max_{i=1}^d |x_i - y_i| > 1$  the random variables  $\omega \mapsto V_\omega(x)$  and  $\omega \mapsto V_\omega(y)$  are independent.

If a Schrödinger operator with random ergodic potential satisfies the above three conditions, the multiscale analysis yields spectral and dynamical localisation in a (small) energy interval containing the bottom of the spectrum  $\inf \sigma(H_\omega)$ , see e.g. [DS99, DBG00, Sto01, GK01, GK04].

Consequently one obtains the following result on localisation.

**Corollary 5** (Localisation). *Let  $H_\omega, \omega \in \Omega$  be a random operator with breather-type potential (1) satisfying Assumptions 1 and 2. Then there is a  $\delta > 0$  such that  $[E_0, E_0 + \delta]$  is a localisation interval for  $H_\omega, \omega \in \Omega$ .*

Let us sketch the strategy of proof of Theorem 4 which is carried out in Section 2.

To establish Lifshitz asymptotics of the integrated density of states we need to analyse the properties of eigenvalues close to the minimum of the spectrum and their associated eigenfunctions. More precisely, we

- (i) reduce the problem to bounds on the position of the lowest eigenvalue of the random Schrödinger operator. This is possible since we are interested only in the leading exponent in the relation (16).
- (ii) identify the configuration of random variables  $\{\lambda_k\}_{k \in \mathbb{Z}^d}$  such that the corresponding potential produces the minimal (with respect to the randomness) ground state energy
- (iii) establish how close a random potential has to be to the minimal potential configuration to produce an eigenvalue  $\mathcal{E}$  close to the overall minimum of the ground state energies.
- (iv) apply a large deviations estimate to bound the probability that such “close by” configurations of the potential occur.

Actually the above steps are implemented for restrictions of the considered Schrödinger operators on finite cubes with increasing length scales  $L \approx \mathcal{E}^{-1/2}$ . Moreover on each scale we have to modify the random variable by a scale dependent cut-off.

## 2. PROOF OF LIFSCHITZ TAILS

Denote by  $\psi_1$  the  $L^2$ -normalised, positive ground state of  $H_{\text{per}}^{1,\text{per}}$  and by  $\Psi$  its periodic extension on the whole of  $\mathbb{R}^d$ . Then  $\psi_L := L^{-d/2}\Psi$  is the normalised ground state of  $H_{\text{per}}^{L,\text{per}}$ . Denote by  $E_1(H_{\text{per}}^{L,\text{per}})$  the lowest eigenvalue of  $H_{\text{per}}^{L,\text{per}}$ . Using singular sequences and the fact that a pointwise positive  $L^2$ -eigenfunction must be the ground state we conclude that  $\min \sigma(H_{\text{per}}) = E_1(H_{\text{per}}^{1,\text{per}}) = E_L(H_{\text{per}}^{L,\text{per}})$  for all  $L \in \mathbb{N}$ .

**Remark 6** (Spectral gap for the periodic operator). Condition (6) ensures that  $\Psi_L$  is also the ground state of  $H_{\text{per}}^{L,N}$  for all  $L \in \mathbb{N}$ . Thus  $\min \sigma(H_{\text{per}}) = E_1(H_{\text{per}}^{1,N}) = E_L(H_{\text{per}}^{L,N})$ . From [KS87] we infer that there exists a constant  $\epsilon_0 > 0$  such that

$$E_2(H_{\text{per}}^{L,N}) - E_1(H_{\text{per}}^{L,N}) \geq \epsilon_0 L^{-2}$$

for all  $L \in \mathbb{N}$ . By adding a constant to the periodic potential we may assume without loss of generality that  $\min \sigma(H_{\text{per}}) = 0$ . Thus

$$(18) \quad E_2(H_{\text{per}}^{L,N}) \geq \epsilon_0 L^{-2} \text{ for all } L \in \mathbb{N}$$

Since the ground state  $\Psi$  is pointwise uniformly bounded away from zero  $c_3 := \inf_{x \in \mathbb{R}^d} \Psi(x) > 0$ . We abbreviate by  $d\alpha(x)$  the measure  $\Psi(x)^2 dx$  on  $\mathbb{R}^d$  and use  $c_4 := \sup_{x \in \mathbb{R}^d} \Psi(x) > 0$ .

Since the dependence of the potential on the coupling constants  $\lambda_k, k$  is not linear we introduce new random variables  $\xi_k, k$ . The scalar product we are interested in can be expressed as a sum in terms of these new random variables.

**Remark 7** (Mapped random variables). Introduce for a parameter  $c_2 \leq \frac{\epsilon_0 \epsilon_1}{2 c_4}$ , i. e.  $\frac{c_2 c_4}{\epsilon_1} \leq \frac{\epsilon_0}{2}$ , the cut-off random variables

$$\tilde{\lambda}_k := \min\{\lambda_k, \lambda_- + c_2 L^{-2}\} \in [\lambda_-, \lambda_- + c_2 L^{-2}]$$

and the non-linearly mapped random variables

$$\xi_k(\omega) = \xi(\lambda_k) := \int d\alpha(x) \tilde{u}(\tilde{\lambda}_k(\omega), x - k)$$

Denote  $\tilde{H}_\omega := H_{\text{per}} + \tilde{V}_\omega$  and  $\tilde{V}_\omega := \sum_{k \in \mathbb{Z}^d} \tilde{u}(\tilde{\lambda}_k(\omega), x - k)$ .

**Remark 8** (Analysis of the first moment). The following expectation value will play a crucial role in the sequel

$$\begin{aligned} (19) \quad \langle \psi_L, \tilde{H}_\omega^{L,N} \psi_L \rangle &= \langle \psi_L, H_{\text{per}}^{L,N} \psi_L \rangle + \langle \psi_L, \tilde{V}_\omega^{L,N} \psi_L \rangle = \langle \psi_L, \tilde{V}_\omega^{L,N} \psi_L \rangle \\ &= L^{-d} \sum_{k \in I_L} \int d\alpha(x) \tilde{u}(\tilde{\lambda}_k(\omega), x - k) = L^{-d} \sum_{k \in I_L} \xi_k(\omega) \end{aligned}$$

For  $c_2 L^{-2} \leq \epsilon_2$ , i. e.  $\tilde{\lambda}_0 \leq \lambda_- + \epsilon_2$ , we have

$$\begin{aligned} \xi(\tilde{\lambda}_0) &= \int d\alpha(x) u(\tilde{\lambda}_0, x) - \int d\alpha(x) u(\lambda_-, x) = \int_{\lambda_-}^{\tilde{\lambda}_0} d\tau \frac{d}{d\tau} \int d\alpha(x) u(\tau, x) \\ &\leq \frac{\tilde{\lambda}_0 - \lambda_-}{\epsilon_1} c_4 \quad \text{by (5)} \\ &\leq \frac{c_2 c_4}{\epsilon_1} \frac{1}{L^2} \quad \text{by definition of } \tilde{\lambda}_0 \end{aligned}$$

Hence

$$\langle \psi_L, \tilde{H}_\omega^{L,N} \psi_L \rangle = L^{-d} \sum_{k \in \mathbb{Z}^d} \xi_k(\omega) \leq \frac{c_2 c_4}{\epsilon_1} \frac{1}{L^2}$$

and thus for  $\nu := \frac{\epsilon_0}{2} L^{-2} + \langle \psi_L, \tilde{H}_\omega^{L,N} \psi_L \rangle$  we have

$$(20) \quad \nu \leq \frac{\epsilon_0}{2} \frac{1}{L^2} + \frac{c_2 c_4}{\epsilon_1} \frac{1}{L^2} \leq \frac{\epsilon_0}{L^2} \leq E_2(H_{\text{per}}^{L,N})$$

by the choice of  $c_2$  and (18).

**Remark 9** (Analysis of the second moment). We will need also an estimate for the second moment. By a zeroth order Taylor expansion one sees that for some  $\hat{\lambda} \in [\lambda_-, \lambda]$

$$\tilde{u}^2(\lambda, x) = 2(\lambda - \lambda_-) \tilde{u}(\hat{\lambda}, x) \frac{\partial \tilde{u}(\hat{\lambda}, x)}{\partial \hat{\lambda}}$$

By (4) we have  $0 \leq \tilde{u}(\hat{\lambda}, x) \leq \tilde{u}(\lambda, x)$  and thus

$$\begin{aligned} \tilde{u}^2(\lambda, x) &\leq 2(\lambda - \lambda_-) \tilde{u}(\lambda, x) \frac{\partial \tilde{u}}{\partial \lambda}(\hat{\lambda}, x) \\ &\leq 2(\lambda - \lambda_-) \tilde{u}(\lambda, x) \sup_{x \in \mathbb{R}^d} \frac{\partial \tilde{u}}{\partial \lambda}(\hat{\lambda}, x) \leq 2\kappa_1(\lambda - \lambda_-) \tilde{u}(\lambda, x) \end{aligned}$$

by Assumption (3). Hence

$$\int d\alpha(x) \tilde{u}^2(\tilde{\lambda}_k, x - k) \leq 2\kappa_1 c_2 L^{-2} \int d\alpha(x) \tilde{u}(\tilde{\lambda}_k, x - k) = 2\kappa_1 c_2 L^{-2} \xi_k$$

and

$$(21) \quad \|\tilde{H}_\omega^{L,N} \psi_L\|^2 = L^{-d} \sum_{k \in I_L} \int d\alpha(x) \tilde{u}^2(\tilde{\lambda}_k, x - k) \leq 2\kappa_1 c_2 L^{-2} L^{-d} \sum_{k \in I_L} \xi_k$$

The next theorem provides us with a lower bound on the first eigenvalue of a random box Hamiltonian. It is formulated in terms of an average of the random variables  $\xi_k, k$ . To prove it we use the Temple-inequality. The bounds on the first and second moment derived in the two preceding remarks are used on one hand to show that Temple's inequality is at all applicable, and on the other hand to insert them into Temple's inequality to obtain an appropriate lower bound.

**Theorem 10.** *Choose  $c_2$  small enough such that  $\frac{4\kappa_1 c_2}{\epsilon_0} < \frac{1}{4}$ . Then*

$$E_1(\tilde{H}_\omega^{L,N}) \geq \frac{3}{4} L^{-d} \sum_{k \in I_L} \xi_k(\omega)$$

*Proof.* To ensure that *Temple's inequality* can be applied to the operator  $\tilde{H}_\omega^{L,N}$  and the vector  $\psi_L$ , we need to establish a chain of inequalities, see for instance Theorem XIII.5 [RS78]. Assume  $c_2 L^{-2} \leq \epsilon_2$ , then

$$\begin{aligned} 0 = E_1(H_{\text{per}}^{L,N}) &\leq E_1(\tilde{H}_\omega^{L,N}) && \text{by monotonicity (4)} \\ &\leq \langle \psi_L, \tilde{H}_\omega^{L,N} \psi_L \rangle && \text{by the min-max Theorem} \\ &< \nu && \text{since } \epsilon_0 > 0 \\ &\leq E_2(H_{\text{per}}^{L,N}) && \text{by inequality (20)} \\ &\leq E_2(\tilde{H}_\omega^{L,N}) && \text{by monotonicity (4)} \end{aligned}$$

We have checked the prerequisites for Temple's inequality and may apply it to the operator  $\tilde{H}_\omega^{L,N}$  and the vector  $\psi_L$ :

$$\begin{aligned} E_1(\tilde{H}_\omega^{L,N}) &\geq \langle \psi_L, \tilde{H}_\omega^{L,N} \psi_L \rangle - \frac{\|\tilde{H}_\omega^{L,N} \psi_L\|^2}{\nu - \langle \psi_L, \tilde{H}_\omega^{L,N} \psi_L \rangle} \\ &\geq \langle \psi_L, \tilde{H}_\omega^{L,N} \psi_L \rangle - \frac{2\kappa_1 c_2 L^{-2} L^{-d} \sum_{k \in I_L} \xi_k(\omega)}{\frac{\epsilon_0}{2} L^{-2}} \\ &\geq \langle \psi_L, \tilde{H}_\omega^{L,N} \psi_L \rangle - \frac{4\kappa_1 c_2}{\epsilon_0} L^{-d} \sum_{k \in I_L} \xi_k(\omega) \end{aligned}$$

Here we used equation (21).

It follows  $E_1(\tilde{H}_\omega^{L,N}) \geq \left(1 - \frac{4\kappa_1 c_2}{\epsilon_0}\right) \langle \psi_L, \tilde{H}_\omega^{L,N} \psi_L \rangle = \left(1 - \frac{4\kappa_1 c_2}{\epsilon_0}\right) \sum_{k \in I_L} \xi_k(\omega)$  and thus we have proven the Theorem.  $\square$

The theorem in turn implies an estimate on how low most of the random variables  $\xi_k, k \in I_L$  are lying, if the principal eigenvalue of  $\tilde{H}_\omega^{L,N}$  is low.

**Corollary 11.** *Let  $\frac{4\kappa_1 c_2}{\epsilon_0} < \frac{1}{4}$ ,  $c_6 > 0$  and set  $\mathcal{E} = \frac{c_6}{L^2} > 0$ . Then we have*

$$E_1(\tilde{H}_\omega^{L,N}) \leq \mathcal{E} \quad \text{implies} \quad \#\{k \in I_L \mid \xi_k < 4\mathcal{E}\} > \frac{L^d}{2}$$

*Proof.* If the conclusion is false then

$$\#\{k \in I_L \mid \xi_k \geq 4\mathcal{E}\} \geq \frac{L^d}{2}$$

Hence  $\sum_{k \in I_L} \xi_k(\omega) \geq 4\mathcal{E} \frac{L^d}{2} = 2\mathcal{E}L^d$ . It follows from Theorem 10

$$E_1(\tilde{H}_\omega^{L,N}) \geq \frac{3}{4}L^{-d} \cdot 2\mathcal{E}L^d = \frac{3}{2}\mathcal{E} > E_1(\tilde{H}_\omega^{L,N})$$

which yields a contradiction.  $\square$

Now we have to show that the event

$$\#\{k \in I_L \mid \xi_k < 4\mathcal{E}\} > \frac{L^d}{2}$$

has an exponentially small probability in the parameter  $L^d$ . To this aim we transform back first to the random variables  $\tilde{\lambda}_k, k$  and then to  $\lambda_k, k$ .

**Lemma 12.** *For  $c_7 \geq 4/\epsilon_1$  and  $c_7\mathcal{E} \leq \epsilon_2$ , i.e.  $\mathcal{E}$  small enough, we have*

$$\xi_k \in [0, 4\mathcal{E}[ \quad \text{implies} \quad \tilde{\lambda}_k \in [\lambda_-, \lambda_- + c_7\mathcal{E}[$$

*Proof.* Assume  $\tilde{\lambda}_k \geq \lambda_- + c_7\mathcal{E}$ . By (5) we have for  $c_7\mathcal{E} \leq \epsilon_2$

$$\begin{aligned} \xi_k(\omega) &\geq \int d\alpha(x)u(\lambda_- + c_7\mathcal{E}, x) - \int d\alpha(x)u(\lambda_-, x) \\ &= \int_{\lambda_-}^{\lambda_- + c_7\mathcal{E}} d\tau \frac{d}{d\tau} \int d\alpha(x)u(\tau, x) \\ &\geq c_7\mathcal{E}\epsilon_1 \\ &\geq 4\mathcal{E} \quad \text{since } c_7 \geq 4/\epsilon_1 \end{aligned}$$

It follows  $\xi_k \geq 4\mathcal{E}$  which is a contradiction.  $\square$

Now choose  $c_6 \leq \frac{c_2}{2c_7}$ . Then  $\tilde{\lambda}_k \in [\lambda_-, \lambda_- + c_7c_6L^{-2}]$  implies  $\lambda_k \in [\lambda_-, \lambda_- + c_7c_6L^{-2}]$ . Thus, for  $c_7 \geq 4/\epsilon_1$ ,  $\frac{c_7c_6}{L^2} \leq \epsilon_2$  and  $c_6 \leq \frac{c_2}{2c_7}$ ,

$$E_1(\tilde{H}_\omega^{L,N}) \leq \frac{c_6}{L^2} \quad \text{implies} \quad \#\{k \in I_L \mid \lambda_k < \lambda_- + c_7c_6L^{-2}\} > \frac{L^d}{2}$$

Now standard large deviations results imply

**Lemma 13.** *Let  $\{\lambda_k\}_{k \in \mathbb{Z}^d}$  be collection of independent, identically distributed random variables on the probability space  $(\Omega, \mathbb{P})$  with  $\mathbb{E}\{\lambda_0\} > \epsilon_3$ . Then there is a constant  $c_8$  such that*

$$\mathbb{P}\{\#\{k \in I_L \mid \lambda_k < \epsilon_3\} > \frac{L^d}{2}\} \leq e^{-c_8L^d}$$

For sufficiently large  $L$  we have  $\mathbb{E}\{\lambda_0\} > \lambda_- + c_7c_6L^{-2}$  and can thus apply the lemma.



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